Asset Pricing & Portfolio Choice
Lecture 21

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Overview

Lecture 21 discusses the efficient set mathematics of CAPM. We finish last lecture's minimization problem to get the efficient CAPM frontier. Following this is a run-through of six propositions given the efficient set mathematics. Furthermore we briefly touch the subject of zero-beta CAPM.
Relevant chapters: 8 (Section 8.4-8.8)

In the following we will not consider the existence of risk free assets.

Minimization problem

Continuing from last lecture we are solving the minimization problem of minimizing the portfolio variance subject to a expected return constraint and a full investment constraint to get the efficient CAPM frontier:

$$\min_{w} \frac{1}{2} w^T \Sigma w \quad \text{st.} \quad w^T \mu = \mu_p, w^T e = 1 \tag{1}$$

We optimize this using the lagrangian to find the optimal portfolio weights:

$$\mathcal{L}(w, \lambda, \gamma) = \frac{1}{2} w^T \Sigma w - \lambda(w^T \mu - \mu_p) - \gamma(w^T e - 1)$$

Solving this gives the following first order condition (FOC):

$$w_p = \lambda \Sigma^{-1} \mu + \gamma \Sigma^{-1} e \quad \text{where} \quad \lambda = \frac{\mu_p C - A}{D} \quad \text{and} \quad \gamma = \frac{B - \mu_p A}{D}$$

We remember that the following scalars A, B, C and D are defined as:

$$\begin{aligned} A &= \mu^T \Sigma^{-1} e \\ B &= \mu^T \Sigma^{-1} \mu \\ C &= e^T \Sigma^{-1} e \\ D &= BC - A^2 \end{aligned}$$

Our objective is to examine the mathematics behind the efficient frontier (INSERT GRAPHIC), where the minimization problem finds each individual point on the frontier for one given number of expected return μ.
Plugging the values in for the lagrangian multipliers λ and γ we end up getting the optimal portfolio weights:

$$w_p = \left(\frac{\mu_p C - A}{D} \right) \Sigma^{-1} \mu + \left(\frac{B - \mu_p A}{D} \right) \Sigma^{-1} e$$

This can be rewritten as:

$$w_p = \frac{1}{D}(B \Sigma^{-1} e - A \Sigma^{-1} \mu) + \frac{1}{D}(C \Sigma^{-1} \mu - A \Sigma^{-1} e) \mu_p$$

From this expression we get that the optimal portfolio weights are given as the sum of two Nx1 vectors. The first term on the RHS is a vector that only needs to be calculated once (which we will denote as g), whereas the second term is a vector times the expected return of the portfolio thus depending on the various levels of the expected return (this term will be denoted as $h\mu_p$). When we know the input values of μ, Σ^{-1}, etc. we can calculate the vectors and we end up with the following expression for the optimal portfolio weights:

$$w_p = g + h\mu_p \quad \text{where} \quad \mu_p = w_p^T \mu, \sigma_p^2 = w_p^T \Sigma w_p$$

Intermezzo - How are the market portfolio weights determined in the real investment universe?

In traditional finance theory, the market portfolio (tangent portfolio) often represents the aggregate of all investable assets. The weights determining this portfolio are defined as "market capitalization weighting". This is a method of determining the weights of individual assets within the market portfolio based on their market values. The weight of each asset is proportional to its market capitalization (market value). Larger companies, in terms of market capitalization, will therefore have a greater influence on the overall market portfolio. Critics of this approach argue that market capitalization weighting can lead to overconcentration in a few large-cap stocks, potentially introducing systemic risk.

Propositions

Given the results of efficient set mathematics follows six propositions.

Proposition 1

For the global minimum variance portfolio we have:

$$w_{MV} = \frac{1}{C}\Sigma^{-1}e$$
$$\mu_{MV} = \frac{A}{C}$$
$$\sigma^2_{MV} = \frac{1}{C}$$

To show the above, we will solve a revised minimization problem of (1) where we do not take into account the expected return of the portfolio:

$$\min_w = \frac{1}{2}w^T\Sigma w \quad \text{st.} \quad w^Te$$

Again we use the lagrangian method:

$$\mathcal{L}(w,\eta) = \frac{1}{2}w^T\Sigma w - \eta(w^Te - 1)$$

FOC:

$\Sigma w_{MV} - \eta e = 0$ we will multiply both sides with the inverse covariance matrix to be able to isolate the weight $\Leftrightarrow$
$\Sigma^{-1}\Sigma w_{MV} - \eta\Sigma^{-1}e = 0 \Leftrightarrow$
$w_{MV} - \eta\Sigma^{-1}e = 0 \Leftrightarrow$
$w_{MV} = \eta\Sigma^{-1}e$

To figure out what the lagrangian multiplier η corresponds to, we use a trick by multiplying both sides with e^T: $e^Tw_{MV} = \eta\underline{e^T\Sigma^{-1}e} = 1$. We know it is equal to 1 given by the previous constraint: $w_{MV}e = e^Tw_{MV} = 1$. The underlined part is the scalar C. We can therefore conclude that: $\eta C = 1 \Leftrightarrow \eta = \frac{1}{C}$.
Substituting this into our expression for $w_{MV} = \eta\Sigma^{-1}e = \frac{1}{C}\Sigma^{-1}e$.

To show that $\mu_{MV} = \frac{A}{C}$ we make use of the first constraint from (1):

$$\mu_{MV} = w^T_{MV}\mu = \frac{1}{C}\underline{e^T\Sigma^{-1}\mu} = \frac{A}{C} \quad \text{where the underlined part equals the scalar A}$$

To show that the variance is given by $\sigma^2_{MV} = \frac{1}{C}$:

$$\sigma^2_{MV} = w^T_{MV}\Sigma w_{MV} = \frac{1}{C}e^T\Sigma^{-1}\Sigma\Sigma^{-1}e\frac{1}{C} = \frac{1}{C}e^TI\Sigma^{-1}e\frac{1}{C} = \frac{1}{C}e^T\Sigma^{-1}e\frac{1}{C} = \frac{C}{C^2} = \frac{1}{C}$$

Proposition 2

Any convex combination of minimum variance frontier portfolios is also a minimum variance frontier portfolio. This proposition turns out to be important for zero beta CAPM.

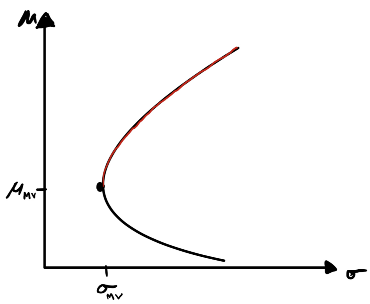
First se recall what a convex combination is:

$$\sum_{i=1}^{K} \alpha_i x_i \quad \text{where} \quad \sum_{i=1}^{K} \alpha_i = 1 \quad \text{and} \quad \alpha_i \geq 0, \forall_i$$

The proof then follows, where α_i represents the weights of different minimum variance efficient portfolios, and w_i the weights of stocks in the individual minimum variance efficient portfolio i:

$$\begin{aligned}\sum_{i=1}^{K} \alpha_i w_i &= \sum_{i=1}^{K} \alpha_i (g + h\mu_i) \\ &= g\sum_{i=1}^{K} \alpha_i + h\sum_{i=1}^{K} \alpha_i \mu_i \\ &= g + h\sum_{i=1}^{K} \alpha_i \mu_i\end{aligned}$$

We can therefore define an efficient portfolio as a minimum variance portfolio with expected return $\geq \mu_{MV}$, which is the entire set on the efficient frontier marked with red.

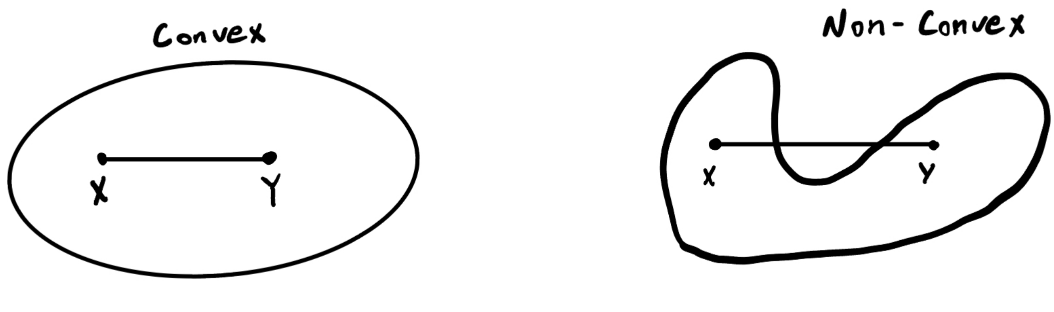


Defined mathematically:

$$\begin{aligned}&\mu_i, ..., \mu_k \geq \mu_{MV} \Rightarrow \\ &\sum_{i=1}^{k} \alpha_i \mu_i \geq \mu_{MV}\end{aligned}$$

Corollary:

The set of efficient portfolios is convex, meaning a convex combination will always be in the convex set:



Proposition 3

Let p and r be any two minimum variance frontier portfolios. The covariance of returns of p and r are given by:

$$Cov(\tilde{r}_p, \tilde{r}_r) = \frac{C}{D}\left(\mu_p - \frac{A}{C}\right)\left(\mu_r - \frac{A}{C}\right) + \frac{1}{C}$$

Proposition 4

The covariance of the return of the global minimum variance portfolio and the return of any other minimum variance portfolio is given by:

$$Cov(\tilde{r}_{MV}, \tilde{r}_r) = \frac{1}{C}$$

We note that this is the same as the variance of the global minimum variance portfolio.
Proof where we make use of proposition 3:

$$cov(\tilde{r}_{MV}, \tilde{r}_r) = \frac{C}{D}\left(\mu_{MV} - \frac{A}{C}\right)\left(\mu_r - \frac{A}{C}\right) + \frac{1}{C} = \frac{1}{C}, \quad \text{as} \quad (\mu_{MV} - \frac{A}{C})(\mu_r - \frac{A}{C}) = 0$$

Proposition 5

The covariance between minimum variance frontier portfolio p and any other portfolio of risky assets r is given by:

$$Cov(\tilde{r}_p, \tilde{r}_r) = \lambda\mu_r + \gamma, \quad \lambda \,\&\gamma \quad \text{as defined earlier}$$

Proof:

We make use of the FOC since p is on the minimum variance frontier:

$$\begin{aligned}
Cov(\tilde{r}_p, \tilde{r}_r) &= w_p^T \Sigma w_r \\
&= (\lambda\mu^T\Sigma^{-1} + \gamma e^T \Sigma^{-1})\Sigma w_r \\
&= \lambda\mu^T w_r + \gamma e^T w_r \\
&= \lambda\mu_r + \gamma \text{ since } e^T w_r = 1 \textit{ by investment constraint}
\end{aligned}$$

Proposition 6

For any minimum variance frontier portfolio p (except global minimum variance portfolio), there exists a minimum variance portfolio with zero covariance with respect to p, denoted z_p.

Proof:

We make use of proposition 3:

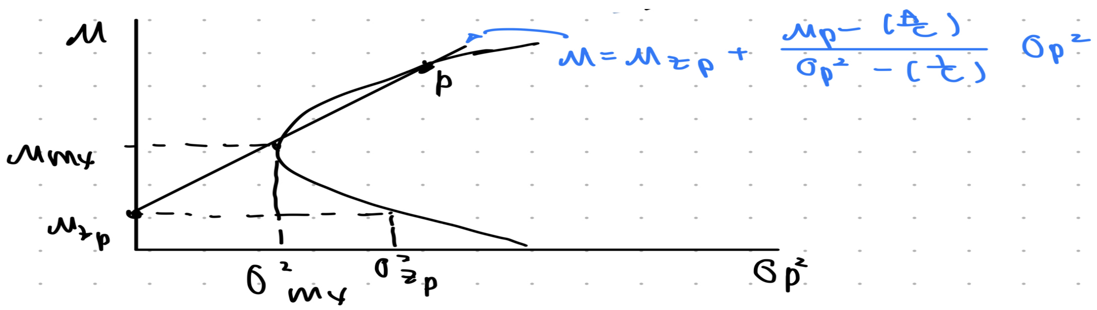
$$Cov(\tilde{r}_p, \tilde{r}_{z_p}) = \frac{C}{D}\left(\mu_p - \frac{A}{C}\right)\left(\mu_{z_p} - \frac{A}{C}\right) + \frac{1}{C} = 0$$

This implies that $\mu_{z_p} = \frac{A}{C} - \frac{\frac{D}{C^2}}{\mu_p - \frac{A}{C}}$

$w_{z_p} = g + h\mu_{z_p}$

$\sigma_{z_p}^2 = w_{z_p}^T \Sigma w_{z_p}$

$\mu = \mu_{z_p} + \frac{\mu_p - \frac{A}{C}}{\sigma_p^2 - \frac{1}{C}}\sigma_p^2$ where this is the equation of the straight line depicted in the graph below



From this we see that the zero covariance portfolio can be found as the intercept on the frontier when moving horizontally from μ_{z_p}.

Then, $Cov(\tilde{r}_p, \tilde{r}_{z_p}) = \lambda\mu_{z_p} + \gamma = 0$ given P.5
This implies that $\gamma = -\lambda\mu_{z_p}$

Therefore, substituting into the previous equation (P.5), $Cov(\tilde{r}_p, \tilde{r}_r) = \lambda\mu_r + \gamma = \lambda(\mu_r - \mu_{z_p})$
This implies that $Cov(\tilde{r}_p, \tilde{r}_p) = \sigma_p^2 = \lambda(\mu_p - \mu_{z_p})$

Dividing $Cov(\tilde{r}_p, \tilde{r}_r)$ with $Cov(\tilde{r}_p, \tilde{r}_p)$ yields

$$\frac{Cov(\tilde{r}_p, \tilde{r}_r)}{\sigma_p^2} = \frac{\lambda(\mu_r - \mu_{z_p})}{\lambda(\mu_p - \mu_{z_p})} \Rightarrow$$

$\beta_{rp}(\mu_p - \mu_{z_p}) = \mu_r - \mu_{z_p}$

where $\beta_{rp} = \dfrac{Cov(\tilde{r}_p, \tilde{r}_r)}{\sigma_p^2}$ measures the volatilty of the portfolio with respect to any MV portfolio.

Rearranging the terms gives: $\mu_r - \mu_{z_p} = \beta_{rp}(\mu_p - \mu_{z_p})$
$\mathbb{E}(\tilde{r}_r) - \mathbb{E}(\tilde{r}_{z_p}) = \beta_{rp}(\mathbb{E}(\tilde{r}_p) - \mathbb{E}(\tilde{r}_{z_p}))$
**Note that the above is a purely mathematical result.*

This leads us to Zero-Beta CAPM where a random minimum variance efficient portfolio, p, is replaced by the market portfolio.

Zero-Beta CAPM

- Same assumptions as CAPM.
- Another additional assumption is assuming that there are no risk-free assets. This imples that all investors are picking portfolios on the minimum-variance frontier. (The choice of portfolio depends on the investor's risk appetite)
- Market portfolio is a convex combination of investor portfolios. This implies that the market portfolio is efficient (by corollary of Proposition 2) and that we know the market portfolio exists and lies on the efficient frontier.
- Thus,

$$\mathbb{E}(\tilde{r}_r) - \mathbb{E}(\tilde{r}_{z_m}) = \beta_{rm}(\mathbb{E}(\tilde{r}_m) - \mathbb{E}(\tilde{r}_{z_m}))$$

- Note that beta of the zero covariance portfolio with respect to market portfolio is zero since covariance is zero. And we can get back to the standard CAPM by replacing $\tilde{r}_{z_m}$ with r_f as we know the risk free rate has zero covariance with any portfolio.